# INTERGRATION:

Integration is a process of obtaining a function from its derivative.

**Techniques of Integration**

**Recognizing the presence of a function of its derivative:**

**Example I**

**Example II**

***Solution***

Let

**Example III**

Let

**Example III**

**Example IV**

***Solution***

Let

**Example VI**

**Example VII**

*.*

Let

***Example VIII***

**Example IX**

**Example X**

***Solution***

Let *u*

*Example XI*

***Solution***

Let

**Students Exercise**

## Integrating trigonometric functions

Considering integration as the reverse process of differentiation. The following examples illustrate the way in which trigonometric functions can be integrated.

|  |  |
| --- | --- |
| *f*(*x*) |  |
| sin *x* |  |
| cos *x* |  |
| sec2x |  |
| sin *ax* |  |
| cosec2*x* |  |
| cot *x* cosec *x* |  |

**Technique II of integration**

**(Integration of product of two cosines) - two sines or a sine and a cosine**

|  |
| --- |
| **The product of two sines, two cosines or a sine and a cosine can be integrated by first expressing the product as a sum or difference of trigonometric functions by use of factor formulae.** |

**Example I**

***Solution***

Consider 

Now compare  with 2cos 3*x* cos *x*

*A + B =* 6*x* ………………………….(i)

A – *B* = 2*x* …………………………….(ii)

Adding Eqn (i) and Eqn (ii);

Eqn (i) – Eqn (ii);

**Example II**

***Solution***

From the factor formulae,

Comparing  with cos 5*x* cos 3*x*

*A + B =* 10*x* ……………………….(i)

*A – B* = 6*x* ……………………….(ii)

Eqn (i) + Eqn (ii)

Eqn (i) – Eqn (ii)



**Example III**

***Solution***

Eqn (i) + Eqn (ii)

Eqn (i) – Eqn (ii)

**Example IV (UNEB 2001)**

***Solution***

Consider

Comparing



*A + B =* 6*x* ………………….. (i)



*A – B =* 2*x* ………………….. (i)

Equation (1) +(2)

2*A =* 8*x*

*A =* 4*x*

Eqn (i) – Eqn (ii);

2*B =* 4*x*

*B =* 2*x*



**Integration of odd and even powers of trigonometric functions**

**Odd powers of trigonometric functions**

Under this we use the following trigonometric identities

**Example I**

Let *u* =

**Example II**

***Solution***

**Example III**

***Solution***

**Example IV**

***Solution***

Let



**Example V**

***Solution***

**TAN & SIN SUBSTITUTION**

|  |
| --- |
| Show that :  **(ii)**  **(ii)**  **(iii)**  **(iv) =** |

***Solution***

Let





**(iii) **

***Solution***

****

****

Let 









**(iv) **

****

Let 





 + *C*



**Tan Substitution**

**Example**

Find the following integrals

***Solution***

Let

***Solution***

Consider

**Example II**

Find the integral of the following.

***Solution***

***Solution***



Let

But cos 2θ = 2cos2*θ* – 1)



4

*x*

**Example III**

Find the integral of the following:

***Solution***

|  |
| --- |
| **Note: For the tan substitution to be used the denominator should not be factorized.** |

By completing squares;

Let 

+C

Consider

By completing squares;

2(*x*2 + 2*x* + 1) – 2 + 11

2(*x* + 1)2 + 9

Let 

Let

**Sine Substitution**

Find the following integrals

***Solution***

Let

***Solution***

=

Let

14

***Solution***

Let =

14

**Example II**

Find the following integrals

***Solution***

Let

Let

But

34

Find the following integrals

***Solution***

Consider

By completing squares;

Let

-2*x*2 + 12*x* – 9 = -2(*x*2 – 6*x*) – 9

By completing squares;

-2(*x*2 – 6*x* + 9) – -18 – 9

-2(*x* – 3)2 + 9

9 – 2(*x* – 3)2

Let

Let

*x* = 4 sin*θ* – 3

*x* + 3

4

**Sec Substitution**

**Note:** When we are integrating integrand in the form. we use the **sec** substitution.

**Example**

Find the following integrals.

***Solution***

1

## Partial Fractions

**Content:**

* Revision of addition and subtraction of rational expressions.
* Expressing rational expressions as a sum of it’s partial fractions
* Rational expression where the denominator has a quadratic term (quadratic factor) which is not factorisable
* Rational expression where the denominator has repeated factors.
* Dealing with improper functions.

**Partial Fractions**

It is a process of expressing a rational expression into simpler rational expression that we can add or subtract to get the original rational expression. Given a rational expression where the numerators are polynomials.

If the degree of the numerator is less than the degree of the denominator the fraction is said to be a proper fractional. If the degree of the numerator is greater or equal to degree of the denominator, the fraction is said to be improper.

Consider 

 can be expressed as a single fraction 

The process of getting back to  from is called expressing  as a partial fraction.

**Methods of Partial Fractions**

**1.** **Denominator with only linear factors**

***Example 1***

Express as a partial fraction

***Solution***

If *x*

If *x* = 1,

*3A* = 3

**Example 2**

Express  as a partial fraction.

*Solution:*

If

*B* = -1

If *x* = 3,

3(3) + 5 = *A*(2 × 3+1)+0

14 = 7*A*

*A =* 2

**Example III**

Express  as partial fraction.

***Solution***

Consider

If *x* = 2,

*B*(1) = 1  *B* = 1

,

**Example IV**

Express  as partial fraction.

***Solution***

******

**

*A*(*x* – 2)(*x* – 3) + *B*(*x* + 1)(*x* – 3) + *C*(*x* + 1)(*x* – 2)

= 3*x*2 – 21*x* + 24

If *x* = 2,

If *x* = 3,

If ,

12A = 48 A = 4

**Example V**

Express  as a partial fraction

***Solution***

If *x*

, *C* = 1

If,

If *x* = 0,

-16*A* = 32 *A =* -2

**Example VI**

Express  in partial fractions.

***Solution***





If *x =* 4, 56*C* = 68 + 44

56*C* = 112  *C* = 2

If *x* = -4, -8*B* = 68 – 44

-8*B* = 24 *B* = -3

If

**Example VII**

Express  as a partial fraction

***Solution***



If ,

If *x* = 1, 4*C* = -8

**Denominator with quadratic factor not factorisable**

**Example I**

Express in partial fractions

***Solution***

; But *A* = 1

**Example II**

Express in partial fractions

***Solution***

If ,

*A*

Equating the corresponding co-efficients

; But *A* = 3

**Example III**

Express  in partial fraction

***Solution***

(*A* + *B*)*x*2 + (*B* – *C*)*x* + 3*A* + *C* = 6 – 3*x*

**Example IV**

Express  in partial fractions

***Solution***

Let

=

*A + C =* 0 ……………………………..(i)

*B + D =* 0……………………………..(ii)

3*A* + 2*C* = 0 ………………………… (iii)

3*B* + 2*D* = 1 …………………………. (iv)

From Eqn (i), *A = -C*

Substitute in Eqn (iii)

From Eqn (ii), *B = -D*

Substitute in Eqn (iv);

**Example IV**

Express  in partial fraction

***Solution***

******

Consider 

Equating coefficients of the same monomial;

; But *A* = 1

1 + *B* = 0  *B* = -1

*A* – *B* + *C* = 2

1 – –1 + *C* = 2 *C* = 0

**Example V**

Express  in partial fractions

***Solution***

If *x* = 4,

3*Ax*2 + 2*Ax* + 3*A* + *Bx*2 – 4*Bx* + *Cx* – 4*C* = 13*x* + 7

3*A* + *B* = 0

*B* = -3*A*



**Example VI**

Express  in partial fractions

***Solution***

If *x* = 2,

*A* + *C* = 0 ……………………………. (i)

(*C* – 2*A* + *B*) = 5 …………………….. (ii)

*C* – 2*B* = 0 …………………………… (iii)

From Eqn (iii); *C* = 2*B*



**Denominator with Repeated Factors**

Express the following in partial fractions.

**Example (Hints)**

Express the following in partial fractions

***Solution***

|  |
| --- |
| **(a)**    **(b)**  **(c)**    **(d)** |

The above hint will help us to express rational expressions with denominators of repeated factors into partial functions

**Example I**

Express  in partial fractions

***Solution***

If , *C*

*C*

If *x* = 0, -*B* = -3 *B* = 3

If *x* = 2, 2*A* + *B* + 4*C* = 2 – 3 – 8

**Example ІІ**

Express  in partial fractions

***Solution***

*A*(*x* – 2)2 + *B*(x – 2)(*x* + 1) + *C*(*x* + 1) = *x* + 4

If ,

*C* = 2

If ,

If ,



**Example III**

Express  in partial fraction.

***Solution***

If ,

If ,

= 3

**Example IV**

Express  in partial fractions

***Solution***



If,

If *x* = -3,

If *x =* 0,



**Example V**

Express  in partial fraction.

***Solution***

*A*(*x* – 1)(*x* + 2) + *B*(*x* + 2) + *C*(*x* – 1)2 = 3*x* + 1

If *x* = 1, 3*B* =

If *x* = -2, 9*C* = -6 + 1

If *x =* 0, -2*A* + 2*B* + *C =* 1



**Example VI**

Express  in partial fractions.

***Solution***



*A*(2*x*–3)(*x* – 4) + *B*(2*x* – 3) + *C*(*x* – 4)2 = 5*x*2 –6*x*–21

If *x* = 4, B(5) = 80 – 24 – 21

If,



If *x* = 0, 12*A* – 3*B* + 16*C* = -21



**Improper Fractions**

So far we have only dealt with proper fractions for which the numerator is of lower degree than the denominator. We can now look at how to deal with improper fractions where the degree of the numerator is greater or equal to the degree of the denominator.

Examples of improper fraction are:





**Example I**

Express  in partial fractions.

***Solution***

If *x* = 0, *A* = 4

If **,

**Example II**

Express  in partial fraction

***Solution***



Consider (*x* – 3)(*x*2 + 1)





*A*(*x*2 + 1) + (*Bx* + *C*)(*x* – 3) = 6*x*2 – 3*x* + 5 ………. (i)

If *x* = 3, 10*A* = 54 – 9 + 5

10*A* = 50

*A* = 5

From Eqn (i);

*Ax*2 + *A* + *Bx*2 – 3*Bx* + *Cx* – 3*C* = 6*x*2 – 3*x* + 5

**Example** **III**

Express  in partial fraction

***Solution***

2

But 

If *x* = 2, 5*A* = 8 – 2 – 1 *A =* 1

**Example IV**

Express  in partial fractions

***Solution***

*A*(*x* – 1)2 + *B*(*x* – 1)(*x* + 2) + *C*(*x* + 2) = 3*x*2 + *x* – 1

If *x* = 1,

*C* = 1

If *x* = -2,

If *x* = 0,



**Example V**

Express  in partial fractions

***Solution***

2*x*3 + 7*x*2 + 2*x* – 10

2*x*3 + 5*x*2 – 3*x*

2*x*2 + 5*x* – 10

2x2 + 5x - 3

2*x*2 + 5*x* – 3

2x2 + 5x - 3

– 7

2x2 + 5x - 3



But

If *x* = -3,

If ,

**Example VI**

Express  in partial fractions.

***Solution***



If *x* = 1, 5*B* = 1 

If *x* = -4, -5*A* = -13 × 4 −12



**Integration of Partial Fraction**

In this section, we are going to look at how we can integrate some algebraic fraction. We will be using partial fractions to express the integrand as a sum of simpler fractions which can be integrated separately. We will also need to call upon wide variety of other techniques including completing squares, integration by substitution, integration using stands results and so on. In order to understand the integration of partial fractions, it’s vital that we undertake a plenty of practice exercise so that they become second nature.

**Note:** It’s important to recognize certain standard integrals and method here.

|  |
| --- |
| **(3)** Splitting up the expression |

Example 1

Consider 

If *x* = 1, 3*A* = 4

If,

**Example II**

***Solution***

Consider,

If *x* = 1, 2*A* = 4

*A* = 2

**Example III**

Consider:



*A*(*x* – 1)(*x* + 5) + *B*(*x* + 5) + *C*(*x* – 1)2 = 36

If *x* = 1,

If *x* = -5,

If *x* = 0,





= 3.1527

**Example IV**

***Solution***

*A*(*x*)(*x*2 + 1) + *B*(*x*2 + 1) + (*Cx* + *D*)*x*2 = 1 + *x*

If *x* = 0, *B* = 1

From Eqn (2)



**Example V**

***Solution***

Consider 



But

If *x* = 2, 4*B* = 8

*B* = 2

If *x* = -2, -4*A* = -8

*A* = 2

**Example VI**

***Solution***

2*x*2 – *x* – 1

12*x*2 – 30*x* – 9

24*x*4 – 72*x*3

24*x*4 – 12*x*3 – 12*x*2

-60*x*3 + 12*x*2

-60*x*3 + 30*x*2 + 30*x*

-18*x*2 - 30*x*

-18*x*2 + 9*x* + 9

-39*x* – 9



But 

If *x =* 1,

If , 





= 500 – 375 – 45) – (256 – 240 – 36) + (16ln4 – 16ln3)

= 100 + 23ln3 -  ln(11) – 32 ln(2)

**Example VII**

***Solution***

Consider.

,

,

,

**Example**



If *x* = 1,

Eqn



Eqn

2B = 1 

From Eqn (2)

**Example IX**

***Solution***

From Eqn (2), *B* = 1 – *D*

Substitute *B* = 1 – *D* in Eqn (4)

From Eqn (1); *A* = -*C*

Substitute in Eqn (3);

Let , 









*A + B =* tan –1(1)

**

**

**

**Example**

Express into partial fractions. Hence evaluate

Consider

If *x* = -1,

If *x* = 2

If *x* = 0,

If *x* = 1, *8A* +B(4)(-1) + *C*(2)(-1) – *D* = 3 + 1 + 1

Eqn (2) – Eqn (1)







**Example (UNEB Question)**

***Solution***

Consider

If *x* = 0,



**Example II**

***Solution***

If *x* = 0, *A* = -1

If *x* = -1, *-C* = 3 – 4 – 1

*C* = 2

If *x* = 1,



**Example (UNEB Question)**

***Solution***

If *x* = -1, -2*B* = 2

*B* = -1

If *x* = -3, 6*c* = 10

If *x* = 0, 3*A* = 1







# DIFFERENTIATION II

Differentiation is a process of finding derivatives

The derivative is the instantaneous rate of change of a function with respect to one of its variables

**Objectives of the topic:**

* To know the derivatives of exponential functions of any base.
* To know the derivatives of logarithmic functions.
* Use the techniques of logarithmic differentiation to find derivatives of functions involving products and quotients.

**Differentiation of exponential functions**

**1. Differentiate the following**

***Solution***

|  |
| --- |
| **Note:**  When we are differentiating an exponential function, we first differentiate the power of the expression multiplied by the same expression. |

***Alternatively***;

*y=* using chain rule

Let *ax*2 *= u*

*y = eu*

*=*

*=*

*= (*

**

**Example II**

Differentiate the following:

1. 
2. 

***Solution***

1. *y =*



1. 



******

******

1. 

****

**Differentiation of logarithmic functions**

|  |
| --- |
| **Note** |

**Example III**

Differentiate the following

***Solution***

*a)*

c)

`

f) 3x (*use product rule*)

**Example I**

Differentiate the following:

***Solution***

**a)**



**Examples**

**Differentiate the following:**

***Solution***

**(a) *y* = ln sin2*x***

**(b) *y* = ln tan(3*x*)**

**(c) *y* = ln 3cos2*x***

**g)**

let

**Example**

Differentiate

***Solution***

Let

**Examples**

Differentiate the following:

a) b) c) d) e) f) g)

***Solution***

a)

)] *dx*

c)

ln *y* = *x*(ln2)

d)

**e)**

Let

**f)**

**g)**

**Differentiation of inverse trigonometric functions**

**Example**

***Solution***

a)



b)



**c)**





*x*

*a*

e)

**Differentiate the following**

c)

d) if show that;

***Solution***



b)

c)

=



**d)**……………………..(1)

From Eqn (1); 

Substituting Eqn (3) in (2)

as required

**Example (UNEB Questions)**

Determine when *x* = 2

(05 *marks*)

***Solution***

Let 





When *x* = 2



**Example (UNEB Question)**

Given that: show that .

***Solution***





Introducing log*e* on both sides,

2ln *y* = ln(1 + sin *x*) − ln(1 − sin *x*)



Substitute for *y*,







Hence  as required

**Example (UNEB Question)**

a) Differentiate the following with respect to *x*

1. (sin *x*)*x*
2. Giving your answers in their simplest forms.

b) The distance of a particle moving in a straight line from a fixed point after time t is given by

*x* = e-tsin *t.*

Show that the particle is instantaneously at rest at time seconds. Find its acceleration at  seconds.

***Solution***

i) Let *y* = (sin *x*)*x*

Introducing loge to both sides,

ln *y =*  *x*ln sin *x*



ii) 

Introducing loge to both sides,



**Example (UNEB Question)**

Given that , show that .

***Solution***

Given *x* = e-t sin *t.*



for instantaneous rest, 

=>  = 0

cos *t* − sin *t* = 0

cos *t* = sin *t*

tan *t* = 1

*t* = tan-1(1)

*t* = seconds

Acceleration = 





When *t* = ,



**Example (UNEB Question)**

a) i) If *x*2sec*x* − *xy* + 2*y*2 = 15, find .

ii) Given that *y* = θ − cos θ; *x* = sin θ; show that.

**b**) Determine the maximum and minimum values of *x*2*e-x*

***Solution***

a) *x*2sec*x* − *xy* + 2*y*2 = 15





ii) *y* = θ − cos θ and *x* = sin θ



Now  (by the chain rule)



Again by using the chain rule,



b) Let *y = x*2*e-x*

By introducing loge on both sides

ln *y* = ln (*x*2*e-x*)

ln *y* = ln *x*2 + ln *e-x*

ln *y* = 2 ln*x* − *x*





For maximum or minimum values of *y*;



⟹ = 0



Either

*x* = 0

Or 2 − *x* = 0

*x* = 2

When *x* = 0, ⟹ *y* = 0

The turning point is (0, 0)

When *x* = 2 ⟹ *y* = 4*e*-2 = 0.5413 (4 dps)

The turning point is (2, 0.5413)

Finding the nature of the turning points





At *x* = 0,  = 2 (*positive*)

Hence the turning point at (0, 0) is a minimum.

Therefore the minimum value of *x*2*e–x* is 0

At *x* = 2,  = 4*e*-2 − 8*e*-2 + 2*e*-2

= -2e-2  (*negative*)

Hence the turning point at (2, 0.5413) is a maximum

Therefore the maximum value of *x*2*e*–*x* is 0.5413

## MACLAURIN’S EXPANSION

|  |
| --- |
| **Maclaurin’s theorem states that:** |

**Example I**

1) Use Maclaurin’s theorem to expand ln(1 + *x*) in ascending powers of x as far as the term x5

*f*V(0) = 24





**2)** Use Mauclaurin’s theorem to expand sec x in ascending powers of x as far as the term ln *x*3

***Solution***



**Example III**

**3(a)** Find the first three terms of the expansion of using Maclaurin’s theorem.

**(b)** Use Maclaurin’s theorem to expand in ascending powers of x up to the term in *x*3

***Solution***

**a)**

*f* '''(*x*) = -6(1 + *x*)-4

*f* '''(*x*) = 

*f* '''(0) = -6



**b)**

Use Maclaurin’s theorem to expand , where a is a constant hence or otherwise expand up to the term ln *x*3

***Solution***





Comparing ln(1 + *x*) with ln(1 + *ax*);

 *a* = 1



Comparing ln(1 – 2*x*) with ln(1 + *ax*);

 *a* = -2

ln(1 – 2*x*) = -2*x* – 2*x*2 – 







**Example IV**

Use Maclaurin’s theorem to show that up to the term in *x*3 is .

Hence evaluate.

***Solution***

)



****

**Differentiation from first principle**

Suppose we have a smooth function *f*(*x*) which is represented graphically by a curve *y* = *f*(*x*) then we can draw the tangent to the curve at any point P. It is important to be able to calculate the slope of the tangent of the curve a graphical method can be used but this is rather imprecise so we use the following analytical method. We choose a second point Q on the curve which is near P and join the two points with a straight line PQ called a secant and calculate the slope of the line. Then, we allow Q to approach P so that the secant swings around until it just touches the curve and becomes a tangent. The limit of the slope of a secant is required to find the slope of a tangent.

Q(*x*+δ*x*, *y*+δ*y*)

P(*x*, *y*)

*y* + δ*y*

*y* = *f*(*x*)

The Greek letter (delta) is used to denote small change (very small change).

In the diagram above figure P(*x*, *y*) and Q(*x*+δ*x*, *y*+δ*y*) are two points on the curve *y* = *f*(*x*). If the increase in *x* in moving from P to Q is then the corresponding increase in *y* is. The coordinates of *Q* are (*x*+). The gradient of the chord

As Q approaches P along the curve ( then becomes zero, PQ coincides with the tangent PT.

Hence, the gradient of the curve at P is the limiting value of

The limiting value of as which is written as and is called derived function. It is not a fraction but a symbol meaning derivative of *y* with respect to *x*.

**Example 1**

Differentiate from the first principles.

Dividing through by

As

Hence

**Example 2**

Differentiate: from the first principles

***Solution***

Divide through by

As

**Example 3**

Differentiate from the first principle

***Solution***

Dividing through by

As

**Example 4**

Differentiate *y* = sin *x* from the first principle.

***Solution***

sin

As

**Example 5**

Differentiate *y* = cos *x* from the first principle

***Solution***

As

As

**Example 6**

Show that from first principles

***Solution***

As

As

**Example 7**

Differentiate from the first principle.

As ,

For small angles,

As

**Example 8**

Differentiate: *y* = sec 3*x* from first principle

As

As

**Example 9**

Differentiate y=sin2x from the first principles

Solution:

As , and

**Example 10**

Differentiate from the first principle

***Solution***

Let A= and B=

As

As

**Example 11**

Differentiate from the first principles

***Solution***

But (*from the tables*)

As

**More examples on differentiation**

**Example I**

Given that , prove that

***Solution***

If then,

**Example II**

If show that

***Solution***

From equation (1)

As required

**Example III**

Show that

***Solution***

From Eqn (1);

…………….. (2)

From Eqn (1)

………. (3)

Substituting Eqn (3) in Eqn (2);

## INTEGRATION BY PARTS

Integration by parts is often used when one has an integral where the integrand can be made to take the form of a product.

When we are integrating by parts, we let the easily differentiable function be U and the easily integrable function to be . However, there are some exceptions.

|  |
| --- |
| **LIATE: Choose U to be a function that comes first in this list (LIATE)**  **L – logarithm function**  **I – Inverse trigonometric functions**  **A – Algebraic function**  **T – Trigonometric function**  **E – exponential functions** |

**Example I**

Since *x* is an algebraic function (A) and cos *x* is a trigonometric function A comes before T in LIATE.

**Example II**

A – comes before E in the word LIATE

Consider

Let

**Example III**

=algebraic function (A)

=trigonometric function (T)

**A** comes first before **T** in the LIATE

**Example IV**

**Example V**

(A)

L come before A in LIATE

+C

**Example VI UNEB 2012**

***Solution***

algebraic function (A)

trigonometric function (T)

A comes before T in the word LIATE

**Example VII**

………….. (i)

Let *u* = *x*2, 

= 2*x*, 

……..(ii)

Let *u* = *x* 

, 

**Example VIII UNEB 2002**

**Example VIII UNEB 2003**

**L** comes before **A** in the word LIATE

**Example IX UNEB 2003**

***Solution***

2*x*2

2*x*2 – 8

8

*x*2 – 4

2





(By partial fractions)

***Alternative method of integration by parts***

If an expression can be broken down into two parts one differentiatable up to zero and the other can be integrated each time the former is differentiated

**Example 1**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | integrate |
| + | *x*2 | cos 2*x* |
| − | 2*x* | sin 2*x* |
| + | 2 | cos 2*x* |
| − | 0 | sin 2*x* |

**Example II**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | integrate |
| + | *x*3 | *e*2*x* |
| − | 3*x*2 | *e*2*x* |
| + | 6*x* | *e*2*x* |
| − | 6 | *e*2*x* |
| + | 0 | *e*2*x* |

**Example III**

***Solution***

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + | *x* | ½ (1+cos 6x) |
| - | 1 |  |
| + | 0 |  |

=

**More examples on integration by parts**

**Example I**

***Solution***

Since (ln *x*) is a logarithmic function **L** and is an algebraic function (A)

L comes before A in LIATES

**Example II**

**Example III**

***Solution***



= 

**Example IV**

***Solution***

(Since )

**Example IV**

**Solution**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + | *u* |  |
| − | 1 |  |
| + | 0 |  |

**Example V**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + | *p* |  |
| − | 1 |  |
| + | 0 |  |

**Example VI**

***Solution***

**Example VII**

**Example VIII**

***Solution***

**Example IX**

**Solution:**

**Cases where the original integral re-appears**

When integrating functions with the original integral re-appearing we use integration by parts. This common with integrals consisting of periodic functions like sin *x* and cos *x*

**Example I**

But the integral on R.H.S is still a product so we can repeat the process

**Example II**

But the integral on RHS still a product so we can repeat the process

**Example III**

But the integral on RHS is still a product so we can repeat the process

However, we can also use the alternative method to integration by parts to evaluate the following integrals

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + |  | cos *x* |
| - |  | -sin *x* |
| + |  | -cos *x* |

*As before*

**Example III**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + |  |  |
| - |  |  |
| + |  |  |

*As before*

**Example IV**

|  |  |  |
| --- | --- | --- |
| Sign change | Differentiate | Integrate |
| + |  |  |
| − |  |  |
| + |  |  |

## Integration of inverse trigonometric functions

**Example I**

***Solution***

*x0* = 1 = algebraic function (A)

‘I’ comes before ‘A’ in the word ***LIATE***

**Example II**

Consider

**Example III**





**Example IV**

**Change of Variable**

The above substitution can be applied to integration of certain trigonometric functions

|  |
| --- |
| Case I  Where the denominator of the variable being integrated is a linear function of the trigonometric function.  e.g.  Where C is a constant  We use the substitution |

|  |
| --- |
| **Case II**  When the expression being integrated is a linear function of the second under trigonometric function e.g.  **Etc**.  We use substitution |

Note

Proof (students’ exercise).

**Example I**

Integrate the following:

***Solution***

If *t* = 1, 2*B* = 1  

If *t* = -1, 2*A* = 1 

If *t* = 2, 4*B* = 2 

If *t* = -2, 4*A* = 2 

***Solution***

Substituting *C* = -*A*, in Eqn (3)

From Eqn (4), *B* = -2*D*

Substituting *B* = -2*D* in Eqn 2

B – 1 = 1

If ,

If,

If,

If,

**Splitting the Numerator**

When a fractional integrand with a quadratic denominator cannot be written in simple partial fractions, it is often to express it as a sum of two fractions by splitting the numerator.

The key to a more general application of this method is to express the numerator in two parts, one of which is a multiple of the derivative of the denominator.

|  |
| --- |
| **Numerator = A(Derivative of denominator) + B** |

**Example**

|  |
| --- |
| **Formula**  Numerator = *A*(derivative of denominator) + *B* |

Equating coefficients of the same monomial;

, 4*A* + *B* = 7

Consider

**Example II**

***Solution***

Numerator = A(Derivative of denominator) + B

But

**Example III**

***Solution***

Consider

**Splitting the numerator for trigonometric functions**

The above method is appropriate to integrals of the form

When splitting the numerator for the trigonometric functions

|  |
| --- |
| **Numerator = A (derivative of the denominator) + B (Denominator)** |

**Example**

From Eqn (i);

**Example II**

***Solution***

Numerator=A(Derivative of the Denominator) + B(Denominator)

…………… (i)

…………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously,



**Example III**

***Solution***

…………… (i)

…………… (ii)

Solving Eqn (i) and Eqn (ii) simultaneously

, 

**Example IV**

Show that

***Solution***



From Eqn

.

= 

## Revision Exercise

1. Express in partial fractions.

(a)  (b) 

(c)  (d) 

(e) 

(f) 

1. Express the following in partial fractions:

(a)  (b) 

(c)  (d) 

(e)  (f) 

1. Express the following in partial fractions:

(a)  (b) 

(c)  (d) 

1. Express in partial fractions

(a)  (b) 

(c)  (d) 

(e)  (f) 

1. Find the following integrals

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i) 

1. Evaluate the following, correct to 3 significant figures.

(a)  (b) 

(c)  (d) 

1. Find the following indefinite integrals

(a)  (b) 

(c)  (d) 

(e)  (f) 

1. Evaluate the following definite integrals

(a)  (b) 

(c)  (d) 

(e)  (f) 

1. Find the following indefinite integrals

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i)  (j) 

1. Find the following definite integrals

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i)  (j) 

1. Find  using integration by parts
2. Find the integral of the following

(a)  (b) 

1. Show that:

(a) 

(b) 

1. Find the following integrals, using the the given change of variable.

(a) ,  = *u*

(b) , = *u*

(c) , 2*x* – 1 = *u*

(d) ,  = *u*

(e) , *x* – 1 = *u*

(f) , *x* – 2 = *u*

(g) , *x* – 2 = *u*

(e) ,  = *u*

1. Find the following integrals using a suitable change of variables only when necessary.

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i)  (j) 

1. Evaluate the following definite integrals by changing the variable and the limits.

(a)  (b) 

(c)  (d) 

(e) 

1. Evaluate the following definite integrals either by writing down the integral as a function of *x* or by using the given change of variable.

(a)  (sec *x* = *u*)

(b)  (cos *x* = *u*)

(c)  (cosec *x* = *u*)

1. Evaluate

(a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i) 

1. Calculate the area enclosed by the curve *y* = , the *x*-axis, *x* = 2 and *x* = 3.
2. Calculate the area under *y* = sin3*x* from *x* = 0 to .
3. Calculate the volume of the solid generated when the area under *y* = cos *x* from *x* = 0 to  is rotated through four right angles about the *x*-axis.
4. The area of a uniform lamina is that enclosed by the curve *y* = sin *x*, the *x*-axis, and the line . Find the distance from the *x*-axis of the centre of gravity of the lamina.

## Answers

1. (a) 

(b)  (c) 

(d)  (e) 

(f) 

2. (a)  (b) 

(c)  (d) 

(e)  (f) 

3. (a) 

(b) 

(c) 

(d) 

4. (a) 

(b) 

(c) 

(d) 

(e) 

(f) 

5. (a)  (b) 

(c)  (d) 

(e)  (f) 

(g) 

(h) 

(i) 

6. (a)  (b) 

(c)  (d) 

7. (a)  (b) 

(c)  (d) 

(e) 

(f) 

8 (a)  (b)  (c) ln 6

(d)  (e)  (f) 

9. (a) sin *x* – *x* cos *x* + *c*

(b) 

(c) -*e-x*(*x* + 1) + *c* (d) 

(e)  (f) 

(g) 

(h) 

(i) 

(j) 

10. (a) -2*x* (b) 

(c)  (d) 1

(e)  (f) 

(g)  (h) 

(i)  (j) 

11. 

12. (a) 

(b) 

14. (a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(g)  (h) 

15. (a)  (b) 

(c)  (d) 

(e)  (f) 

(g)  (h) 

(i)  (j) 

16. (a)  (b)  (c) 

(d)  (e) 

17. (a)  (b)  (c) 

18. (a)  (b)  (c) 

(d)  (e)  (f) 24.3

(g)  (h)  (i) 

19.  20.  21.  22. 